BSE PROPERTY FOR SOME CERTAIN SEGAL ALGEBRAS WITH APPLICATIONS TO THE FOURIER ALGEBRA

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ABSTRACT. In this paper, we study the BSE-property for some certain Segal algebras. As an application, we give some results on subalgebras of the Fourier algebra and provide a wide range of Banach algebras with the BSE-property. Also, we give a generalization of a result due to E. Kaniuth and A. Ülger.

1. INTRODUCTION AND PRELIMINARIES

Let X be a non-empty locally compact Hausdoroff space. A subalgebra A of $C_0(X)$ is called a Banach function algebra if A separates strongly the points of X and with a norm $||\cdot||$, $(A, ||\cdot||)$ is a Banach algebra. We know that for each $f \in A$,

$$||f||_X = \sup\{|f(x)| : x \in X\} \leq ||f||,$$

because for each $f \in A$ and $x \in X$, $|f(x)| = |\phi_x(f)| \leq ||\phi_x|| ||f|| \leq ||f||$ where ϕ_x denotes the evaluation functional. So, $||f||_X \leq ||f||$.

Let G be a locally compcat Abelian group. A subspace $\mathcal{S}$ of $L^1(G)$ is called a (Reiter) Segal algebra if it satisfies the following conditions:

(1) $\mathcal{S}$ is dense in $L^1(G)$.
(2) $\mathcal{S}$ is a Banach space under some norm such that $||f||_1 \leq ||f||_{\mathcal{S}}$ for each $f \in \mathcal{S}$.
(3) f_y is in $\mathcal{S}$ and $||f||_{\mathcal{S}} = ||f_y||_{\mathcal{S}}$ for all $f \in \mathcal{S}$ and $y \in G$ where $f_y(x) = f(y^{-1}x)$.
(4) For all $f \in \mathcal{S}$, the mapping $y \longrightarrow f_y$ is continuous.

The first example of a Segal algebra was given by Wiener in the case $G = \mathbb{R}$. Then for the first time, Segal gave the axioms of Segal alegbars. On the other hand, Reiter improved the axioms of Segal and gave the above definition of a Segal algebra. In [1], Burnham with changing $L^1(G)$ with an arbitrary Banach algebra A gave a generalization of Segal algebras and introduced the notion of an abstract Segal algebra.

It is well-known that $L^1(G)$ is a commutative semi-simple regular Banach algebra with a bounded approximate identity with compact support; see [6].

Recently, Inoue and Takahashi in [4] investigated abstract Segal algebras in a non-unital commutative semi-simple regular Banach algebra B such that B has a bounded approximate identity in B_c where

$$B_c = \{x \in B : \widehat{x} \text{ has compact support}\}.$$

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Indeed, they gave the following definition of a Segal algebra in B.

Definition 1.1. An ideal $\mathcal{S}$ in B is called a Segal algebra in B if $\mathcal{S}$ satisfies the following properties:

(1) $\mathcal{S}$ is dense in B.
(2) $\mathcal{S}$ is a Banach space under some norm $||\cdot||_{\mathcal{S}}$ such that $||a||_B \leq ||a||_{\mathcal{S}}$ for each $a \in \mathcal{S}$.
(3) $||ax||_{\mathcal{S}} \leq ||a||_B ||x||_{\mathcal{S}}$ for each $a \in B$ and $x \in \mathcal{S}$.
(4) $\mathcal{S}$ has approximate units.

Clearly, an abstract Segal algebra in B (in the sense of Burnham) is a Segal algebra in B if and only if it possesses approximate units.

A linear operator T on A is called a multiplier if it satisfies $xT(y) = T(x)y$ for all $x, y \in A$. Suppose that $\mathcal{M}(A)$ denotes the space of all multiplier of the Banach algebra A which is a unital commutative Banach algebra. If $\Delta(A)$ denotes the space of all characters of A, that is, non-zero homomorphisms from A into $\mathbb{C}$, for each $T \in \mathcal{M}(A)$, there exists a unique bounded continuous function $\widehat{T}$ on $\Delta(A)$ such that $\widehat{T(a)}(\phi) = \widehat{T}(\phi)\widehat{a}(\phi)$ for all $a \in A$ and $\phi \in \Delta(A)$; see [8, Theorem 1.2.2]. Let $\widehat{\mathcal{M}(A)}$ denote the space of all $\widehat{T}$ corresponding to $T \in \mathcal{M}(A)$.

A bounded continuous function σ on $\Delta(A)$ is called a BSE-function if there exists a constant $C > 0$ such that for each $\phi_1, ..., \phi_n \in \Delta(A)$ and complex numbers $c_1, ..., c_n$, the inequality

$$|\sum_{i=1}^{n} c_i \sigma(\phi_i)| \leq C||\sum_{i=1}^{n} c_i \phi_i||_{A^*}$$

holds. Let $C_{\mathrm{BSE}}(\Delta(A))$ be the set of all BSE-functions. The Banach algebra A is called a BSE-algebra if $C_{\mathrm{BSE}}(\Delta(A)) = \widehat{\mathcal{M}(A)}$.

The theory of BSE-algebras for the first time introduced and investigated by Takahasi and Hatori; see [10].

In the case that $\Delta(A) \neq \emptyset$, a net $\{a_\alpha\}$ in A is called a bounded Δ-weak approximate identity (b.Δ-w.a.i) for A if $\{a_\alpha\}$ is bounded in A and

$$\lim_{\alpha} \phi(a_\alpha a) = \phi(a) \quad (\phi \in \Delta(A), a \in A),$$

or equivalently, $\lim_\alpha \phi(a_\alpha) = 1$ for each $\phi \in \Delta(A)$. Clearly, each b.a.i for A is a b.Δ-w.a.i and the converse is not valid in general; see [5]. But if A is a Banach function algebra such that A is an ideal in its second dual and A has a b.Δ-w.a.i, then A has a b.a.i; see [2, Proposition 3.1].

In the sequel, we suppose that $(A, ||\cdot||)$ is a natural regular Banach function algebra on a locally compact, non-compact Hausdorff space X with a bounded approximate identity $\{e_\alpha\}$ in A_c.

We will prove that if A is a BSE-algebra and $\tau \in A_{\mathrm{loc}}$, then $A_{\tau(n)}$ is a BSE-algebra if and only if τ is bounded. As an application of this theorem, we prove that if G is a locally compact group and $\tau \in A(G)_{\mathrm{loc}}$, then $A(G)_{\tau(n)}$ is a BSE-algebra if and only if G is amenable. This later theorem, gives a generalization of a theorem due to E. Kaniuth and A. Ülger.

2. A characterization of BSE Segal algebras

We start this section with recalling the following definitions from [4].

Definition 2.1. A complex-valued continuous function σ on X is called a local A-function if for all $f \in A_c$, $f\sigma \in A$. Following the notation of Inoue and Takahashi, A_{loc} denotes the set of all local A-function.

Definition 2.2. For positive integer n and a continuous complex-valued function τ on A, put

$$A_{\tau(n)} = \{f \in A : f\tau^k \in A \quad (0 \leq k \leq n)\},$$

$$||f||_{\tau(n)} = \sum_{k=0}^{n} ||f\tau^k||.$$

Lemma 2.3. *$(A_{\tau(n)}, ||\cdot||_{\tau(n)})$ is a Banach algebra.*

Proof. Clearly, $A_{\tau(n)}$ is a linear subspace of A and $||\cdot||_{\tau(n)}$ is a norm on $A_{\tau(n)}$. For each $f, g \in A_{\tau(n)}$, we have

$$||fg||_{\tau(n)} = \sum_{k=0}^{n} ||(fg)\tau^k|| \leq ||f|| \sum_{k=0}^{n} ||g\tau^k|| \leq ||f||_{\tau(n)} ||g||_{\tau(n)}.$$

Completeness of the norm, follows from the proof of [4, Theorems 5.4]. □

By [4, Theorems 5.4], if $\tau \in A_{\text{loc}}$, then $(A_{\tau(n)}, ||\cdot||_{\tau(n)})$ is a Segal algebra in A such that $\Delta(A_{\tau(n)}) = \Delta(A) = X$, that is, $x \longrightarrow \phi_x$ is a homeomorphism from X onto $\Delta(A_{\tau(n)})$.

Therefore, we can consider $A_{\tau(n)}$ in two ways; first, as a Segal algebra in A and second, as a Banach algebra (when A does not have a b.a.i).

The following theorem is one of our main results.

Theorem 2.4. *Let A be a* BSE-*algebra and $\tau \in A_{\text{loc}}$. Then $A_{\tau(n)}$ is a* BSE-*algebra if and only if τ is bounded.*

Proof. Let $A_{\tau(n)}$ be a BSE-algebra. Then by [10, Corollary 5], $A_{\tau(n)}$ has a b.Δ-w.a.i $\{f_\alpha\}$. Therefore, there exists an $M > 0$ such that

$$||f_\alpha||_{\tau(n)} < M, \quad \lim_\alpha f_\alpha(x) = 1 \qquad (x \in X).$$

But, $||f_\alpha \tau||_X \leq ||f_\alpha \tau|| \leq ||f_\alpha||_{\tau(n)}$ and hence we have

$$|\tau(x)| \leq M \quad (x \in X).$$

Conversely, let τ be bounded by $M > 0$. Since $\tau \in A_{\text{loc}}$, for each $f \in A_c$ by the definition and using this fact that $\text{supp}(f\tau) \subseteq \text{supp}(f)$, we have $f\tau^k \in A$ for $0 \leq k \leq n$. Therefore, $A_c \subseteq A_{\tau(n)}$. Since A has a bounded approximate identity $\{e_\alpha\}$ in A_c, hence A is Tauberian (that is, A_c is dense in A). So, $A_{\tau(n)}$ is dense in A.

Now, we show that the norms of A and $A_{\tau(n)}$ are equivalent and this implies that $A = A_{\tau(n)}$, hence $A_{\tau(n)}$ is a BSE-algebra as A. But for each $f \in A$, we have

$$||f|| \leq ||f||_{A_{\tau(n)}} \leq ||f||(\sum_{k=0}^{n} M^k),$$

which completes the proof. □

By [4, Theorem 9.10], we know that if $\mathcal{S}$ is a Segal algebra in A, then $\mathcal{S}$ is a BSE-algebra if and only if it has a b.Δ-w.a.i. On the other hand, by [7, Theorem 3.1], if $\mathfrak{A}$ is a semi-simple commutative Banach algebra which is an ideal in its second dual, then $\mathfrak{A}$ is a BSE-algebra if and only if $\mathfrak{A}$ has a b.Δ-w.a.i. So, in view of this fact that a natural Banach function algebra is semi-simple, we have the following result.

Proposition 2.5. *Let A be an ideal in A^{**} and $\tau \in A_{\text{loc}}$. Then $A_{\tau(n)}$ is a* BSE-*algebra if and only if it has a b.Δ-w.a.i.*

3. Applications to the Fourier algebra

Let G be a locally compact group. If $1 < p < \infty$, we let $A_p(G)$ denote the subspace of $C_0(G)$ consisting of all functions of the form $u = \sum_{i=1}^{\infty} f_i * \widetilde{g_i}$ where "$*$" denotes the convolution product, $f_i \in L^p(G)$, $g_i \in L^q(G)$, $1/p + 1/q = 1$, $\sum_{i=1}^{\infty} ||f_i||_p ||g_i||_q < \infty$ and $\widetilde{f}(x) = \overline{f(x^{-1})}$ for all $x \in G$. With the pointwise operation and the following norm

$$||u||_{A_p(G)} = \inf\{\sum_{i=1}^{\infty} ||f_i||_p ||g_i||_q : u = \sum_{i=1}^{\infty} f_i * \widetilde{g_i}\},$$

$A_p(G)$ is a Banach algebra called the Figà-Talamanca-Herz algebra.

Also, we know that $\Delta(A_p(G)) = G$, that is, each character of $A_p(G)$ is an evaluation function at some $x \in G$. By [3, Proposition 3], $A_p(G)$ is a regular Banach algebra. If $p = 2$, $A_2(G) = A(G)$ is called the Fourier algebra; see [3] and [6, Section 2.9].

Recall that a group G is said to be amenable if there exists an $m \in L^\infty(G)^*$ such that $m \geq 0$, $m(1) = 1$ and $m(f_x) = m(f)$ for each $x \in G$ and $f \in L^\infty(G)$.

Kaniuth and Ülger proved that $A(G)$ is a BSE-algebra if and only if G is amenable; see [7, Theorem 5.1].

Now, we give the following result which is a generalized version of the mentioned Theorem. Recall that $B(G)$ denotes the Fourier-Stieltjes algebra, consisting of all finite linear combinations of continuous positive definite functions on G; see [9, Section 19].

Theorem 3.1. *Let G be a locally compact group and $\tau \in (A(G))_{\text{loc}}$. Then $A(G)_{\tau(n)}$ is a* BSE-*algebra if and only if G is amenable.*

Proof. Suppose that G is amenable and $\tau \in (A(G))_{\text{loc}}$. So, τ is in $\mathcal{M}(A(G))$ (indeed M_τ defined by $M_\tau v = uv$ is in $\mathcal{M}(A(G))$ and $||M_\tau|| = ||\tau||$). Since G is amenable, by [9, Corollary 19.2] we have $\mathcal{M}(A(G)) = B(G)$. Therefore, $\tau \in B(G)$ and it is well-known that each function in $B(G)$ is bounded. So, by Theorem 2.4 and [7, Theorem 5.1] we conclude that $A(G)_{\tau(n)}$ is a BSE-algebra.

Conversely, let $A(G)_{\tau(n)}$ be a BSE-algebra and $\{f_\alpha\}$ be a b.Δ-w.a.i for $A(G)_{\tau(n)}$. Since, $\Delta(A(G)_{\tau(n)}) = G = \Delta(A(G))$ and $||f_\alpha||_{A(G)} \leq ||f_\alpha||_{\tau(n)}$, therefore $\{f_\alpha\}$ is a b.Δ-w.a.i for $A(G)$. Hence, by [7, Theorem 5.1] G is amenable. □

By using the above theorem and using this fact that $A(G)$ is an ideal in $B(G)$, we can provide a wide range of BSE-algebras as follows.

Corollary 3.2. *If G is amenable and $\tau \in B(G)$, then $A(G)_{\tau(n)}$ is a* BSE-*algebra.*

For the generalized Fourier algebra, that is, the Figà-Talamanca-Herz algebra we have the following result.

Theorem 3.3. *Let G be a locally compact group, $1 < p < \infty$ and $\tau \in (A_p(G))_{\text{loc}}$. If $A_p(G)_{\tau(n)}$ is a* BSE-*algebra, then $A_p(G)_{\tau(n)}$ has a b.Δ-w.a.i and G is amenable. Conversely, if G is discrete and $A_p(G)_{\tau(n)}$ has a b.Δ-w.a.i, then $A_p(G)_{\tau(n)}$ is a* BSE-*algebra.*

Proof. If $A_p(G)_{\tau(n)}$ is a BSE-algebra, then $A_p(G)_{\tau(n)}$ has a b.Δ-w.a.i and therefore G is amenable.

Conversely, suppose that G is discrete and $A_p(G)_{\tau(n)}$ has a b.Δ-w.a.i. Since $A_p(G)$ is a semi-simple commutative and Tauberian Banach algebra and G is discrete, by [7, Remark 3.5], $A_p(G)$ is an ideal in its second dual. Since, $A_p(G)_{\tau(n)}$ has a b.Δ-w.a.i, $A_p(G)$ also has a b.Δ-w.a.i. So, $A_p(G)$ has a b.a.i. Therefore, $A_p(G)_{\tau(n)}$ is a Segal algebra in $A_p(G)$. Now, the result follows by using Proposition 2.5. □

Corollary 3.4. *If G is not amenable, then $A_p(G)_{\tau(n)}$ is not a* BSE-*algebra.*

References

[1] J. T. Burnham. Closed ideals in subalgebras of Banach algebras I. *Proc. Amer. Math. Soc.*, 32:551–555, 1972.

[2] H. G. Dales and A. Ülger. Approximate identities in Banach function algebras. *Studia Mathematica*. To appear.

[3] C. S. Herz. Harmonic synthesis for subgroups. *Annales de l'institut Fourier*, 23(3):91–123, 1973.

[4] J. Inoue and S-E. Takahasi. Segal algebras in commutative Banach algebras. *Rocky Mountain J. Math.*, 44(2):539–589, 2014.

[5] C. A. Jones and C. D. Lahr. Weak and norm approximate identities are different. *Pacific J. Math*, 72(1):99–104, 1977.

[6] E. Kaniuth. *A Course in Commutative Banach Algebras.* Springer Verlag, Graduate texts in mathematics, 2009.

[7] E. Kaniuth and A. Ülger. The Bochner-Schoenberg-Eberlian property for commutative Banach algebras, especially Fourier and Fourier Stieltjes algebras. *Proc. Amer. Math. Soc.*, 362:4331–4356, 2010.

[8] R. Larsen. *the multiplier problem*, volume 105 of *lecture Notes in Math.* Springer-Verlag, New York, 1969.

[9] J. P. Pier. *Amenable locally compact groups*. Wiley Interscience, New York, 1984.

[10] S. E. Takahasi and O. Hatori. Commutative Banach algebras which satisfy a Bochner-Schoenberg-Eberlein-type theorem. *Proc. Amer. Math. Soc.*, 110:149–158, 1990.

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